ON THE STABILITY OF THE MOTION OF A VISCOUS FLUID

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A criterion for the stability of the steady motion of a viscous fluid between two parallel planes was established by Reynolds[1] as $2\rho b U_m/\mu < 517$, where ρ = density, $2b$ = distance between the parallel planes, U_m = mean velocity, and μ = coefficient of viscosity. A much lower figure 167 was found by Sharpe[2] and later Orr[3] obtained the figure 117. In this paper by expressing the small motion which is superimposed on the steady motion (the Reynold's method of solution) in the form of a Fourier's Series it is possible to show that the method employed by Reynolds gives a result as small as that of Orr. Also by applying Calculus of Variations the minimum value 116.8 is obtained.

Equations of Motion.—Consider a viscous fluid of density ρ moving in two dimensions with component velocities u, v, in directions x, y, respectively. The dynamical equations of motion are

$$\rho\frac{\partial u}{\partial t} = -\left\{\frac{\partial}{\partial x}(p_{xx} + \rho uu) + \frac{\partial}{\partial y}(p_{yx} + \rho uv)\right\} \tag{1}$$

$$\rho\frac{\partial v}{\partial t} = -\left\{\frac{\partial}{\partial x}(p_{xy} + \rho vu) + \frac{\partial}{\partial y}(p_{yy} + \rho vv)\right\},$$

where the stresses in the fluid are given by

$$p_{xx} = p - 2\mu\frac{\partial u}{\partial x},\quad p_{yy} = p - 2\mu\frac{\partial v}{\partial y},\quad p_{xy} = p_{yx} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right). \tag{1a}$$

We have also the equation of continuity $\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0.$

Steady Motion.—In the case of a fluid moving between two parallel planes there exists a solution of these equations of the form

$$v = 0,\ u = \frac{1}{\mu}\frac{\partial p}{\partial x}\frac{y^2 - b^2}{2}, \tag{2}$$

where $y = \pm b$ are the equations of the planes and $\dfrac{\partial p}{\partial x}$ is constant. The equations (1) for this case reduce to

$$\frac{\partial p}{\partial x} = \mu\frac{d^2u}{dy^2}. \tag{2a}$$

The average velocity is

$$U_m = \frac{\int_a^b u dy}{b} = -\frac{1}{3\mu}\frac{\partial p}{\partial x} b^2,$$

therefore $\dfrac{\partial p}{\partial x} = -3\mu U_m/b^2, \quad u = \dfrac{3}{2} U_m(b^2 - y^2)/b^2.$

Turbulent Motion.—Consider a motion given by $u = \bar{u} + u'$ and $v = \bar{v} + v'$ where u, v are periodic functions of period l with regards to x and also functions of y such that

$$\int_0^l u'dx = \int_0^l v'dx = 0 \text{ and } \bar{u} = \int_0^l udx,\ \bar{v} = \int_0^l vdx.$$

To obtain the equations of turbulent motion substitute for u, v, in equations (1) and take the mean value of each member over a length l in the x-direction. The resulting equations for the mean motion are

$$\left.\begin{aligned} \rho\frac{\partial \bar{u}}{\partial t} &= -\left\{\frac{\partial}{\partial x}(\bar{p}_{xx} + \rho\bar{u}\bar{u} + \rho\overline{u'u'}) + \frac{\partial}{\partial y}(\bar{p}_{yx} + \rho\bar{u}\bar{v} + \rho\overline{u'u'})\right\} \\ \rho\frac{\partial \bar{v}}{\partial t} &= -\left\{\frac{\partial}{\partial x}(\bar{p}_{xy} + \rho\bar{v}\bar{u} + \rho\overline{v'u'}) + \frac{\partial}{\partial y}(\bar{p}_{yy} + \rho\bar{v}\bar{v} + \rho\overline{v'v'})\right\}, \end{aligned}\right\} \quad (3)$$

where $p_{ij} = \bar{p}_{ij} + p'_{ij}$. Subtracting equations (3) from (1) we have the equations of relative motion,

$$\left.\begin{aligned} \rho\frac{\partial u'}{\partial t} = -\Big\{&\frac{\partial}{\partial x}[p'_{xx} + \rho(\bar{u}u' + u'\bar{u}) + \rho(u'u' - \overline{u'u'})] + \\ &\frac{\partial}{\partial y}[p'_{yx} + \rho(\bar{u}v' + u'\bar{v}) + \rho(u'v' - \overline{u'v'})]\Big\} \\ \rho\frac{\partial v'}{\partial t} = -\Big\{&\frac{\partial}{\partial x}[p'_{xy} + \rho(\bar{v}u' + v'\bar{u}) + \rho(v'u' - \overline{v'u'})] + \\ &\frac{\partial}{\partial y}[p'_{yy} + \rho(\bar{v}v' + v'\bar{v}) + \rho(v'v' - \overline{v'v'})]\Big\}. \end{aligned}\right\} \quad (4)$$

To obtain the rate of change of the energy for the mean motion multiply equation (3) by $\bar{u}, \bar{v}$ and add. Putting $2\bar{E} = \rho(\bar{u}^2 + \bar{v}^2)$ we obtain

$$\frac{\partial \bar{E}}{\partial t} = -\left\{\begin{aligned} &\frac{\partial}{\partial x}[\bar{u}(\bar{p}_{xx} + \rho\overline{u'u'})] + \frac{\partial}{\partial y}[\bar{u}(\bar{p}_{yx} + \rho\overline{u'v'})] \\ &\frac{\partial}{\partial x}[\bar{v}(\bar{p}_{xy} + \rho\overline{v'u'})] + \frac{\partial}{\partial y}[\bar{v}(\bar{p}_{yy} + \rho\overline{v'v'})] \end{aligned}\right\} +$$

$$\left\{\begin{matrix} \bar{p}_{xx}\dfrac{\partial \bar{u}}{\partial x} + \bar{p}_{yx}\dfrac{\partial \bar{u}}{\partial y} \\ \bar{p}_{xy}\dfrac{\partial \bar{v}}{\partial x} + \bar{p}_{xy}\dfrac{\partial \bar{v}}{\partial y} \end{matrix}\right\} + \rho \left\{\begin{matrix} \overline{u'u'}\dfrac{\partial \bar{u}}{\partial x} + \overline{u'v'}\dfrac{\partial \bar{u}}{\partial y} \\ \overline{v'u'}\dfrac{\partial \bar{v}}{\partial x} + \overline{v'v'}\dfrac{\partial \bar{v}}{\partial y} \end{matrix}\right\}. \tag{5}$$

In a similar manner to obtain the equation for the rate of change of mean energy of the relative motion multiply (4) by u', v', and add. Omitting terms whose average value is zero and putting $2\bar{E}' = \rho(\overline{u'u'} + \overline{v'v'})$ we have

$$\frac{\partial \bar{E}'}{\partial t} = -\left\{\begin{matrix} \dfrac{\partial}{\partial x}[u'(p'_{xx} + \rho u'u') + \dfrac{\partial}{\partial y}[u'(p'_{yx} + \rho u'v')] \\ \dfrac{\partial}{\partial x}[v'(p'_{xy} + \rho v'u') + \dfrac{\partial}{\partial y}[v'(p_{yy} + \rho v'v')] \end{matrix}\right\} +$$

$$\left\{\begin{matrix} p'_{xx}\dfrac{\partial u'}{\partial x} + p_{yx}\dfrac{\partial u'}{\partial y} \\ p'_{xy}\dfrac{\partial v'}{\partial x} + p_{yy}\dfrac{\partial v'}{\partial y} \end{matrix}\right\} - \rho \left\{\begin{matrix} \overline{u'u'}\dfrac{\partial \bar{u}}{\partial x} + \overline{u'v'}\dfrac{\partial \bar{u}}{\partial y} \\ \overline{v'u'}\dfrac{\partial \bar{v}}{\partial x} + \overline{v'v'}\dfrac{\partial \bar{v}}{\partial y} \end{matrix}\right\}. \tag{6}$$

If we integrate (5) and (6) between the planes, the last terms in each gives the conversion of energy of mean motion into energy of relative motion and is the same except for sign. After substituting the value from (1a) we have from (6)

$$\int_{-b}^{b}\int_{0}^{l}\frac{d\bar{E}'}{dt}\,dxdy = -\rho\int_{-b}^{b}\int_{0}^{l}\left\{\begin{matrix} \overline{u'u'}\dfrac{\partial \bar{u}}{\partial x} + \overline{u'v'}\dfrac{\partial \bar{u}}{\partial y} \\ \overline{v'u'}\dfrac{\partial \bar{v}}{\partial x} + \overline{v'v'}\dfrac{\partial \bar{v}}{\partial y} \end{matrix}\right\} dxdy$$
$$-\mu\int_{-b}^{b}\int_{0}^{l}\left\{\begin{matrix} 2\left[\left(\dfrac{\partial u'}{\partial x}\right)^2 + \left(\dfrac{\partial v'}{\partial y}\right)^2\right] \\ + \left(\dfrac{\partial v'}{\partial x} + \dfrac{\partial u'}{\partial y}\right)^2 \end{matrix}\right\} dxdy. \tag{7}$$

In order that the relative motion may not die out it is necessary that the right-hand member of (7) be greater than or equal to zero, or

$$\int_{-b}^{b}\int_{0}^{l}\frac{d\bar{E}'}{dt}\,dxdy = \rho I_1 - \mu I_2 \geq 0, \tag{8}$$

where I_1, I_2, are the double integrals in equation (7). Hence, values of u', v' must exist for which I_2/I_1, must be $\geq \rho/\mu$ for a minimum value. They must also satisfy the continuity equations $\partial u'/\partial x + \partial v'/\partial y = 0$ and the boundary conditions $u' = v' = 0$ when $y = \pm b$.

Assuming
$$\begin{cases} u' = \dfrac{d\alpha}{dy}\cos\dfrac{2\pi x}{l} + \dfrac{d\beta}{dy}\sin\dfrac{2\pi x}{l} \\ v' = \dfrac{2\pi}{l}\,\alpha\sin\dfrac{2\pi x}{l} - \dfrac{2\pi}{l}\beta\cos\dfrac{2\pi x}{l} \end{cases}$$

where $\alpha = \beta = \dfrac{d\alpha}{dy} = \dfrac{d\beta}{dy} = 0$, when $y = \pm b$, substituting for u' and v' in I_1, I_2 and integrating over the length l in the x-direction we have

$$I_1 = \frac{l}{2}\int_{-b}^{b}\left\{\frac{2\pi}{l}\left(\alpha\frac{d\beta}{dy} - \beta\frac{d\alpha}{dy}\right)\frac{d\bar{u}}{dy}\right\}dy, \qquad (8a)$$

$$I_2 = \frac{l}{2}\int_{-b}^{b}\left\{\left(\frac{2\pi}{l}\right)^4(\alpha^2+\beta^2) + 2\left(\frac{2\pi}{l}\right)^2\left[\left(\frac{d\alpha}{dy}\right)^2 + \left(\frac{d\beta}{dy}\right)^2\right] + \left(\frac{d^2\alpha}{dy^2}\right)^2 + \left(\frac{d^2\beta}{dy^2}\right)^2\right\}dy. \qquad (9)$$

From equation (3) we have $\dfrac{dp}{dx} = \mu\left(\dfrac{\partial^2\bar{u}}{\partial y^2}\right) - \rho\dfrac{d}{dy}\left(\overline{u'v'}\right)$. Comparing this with (2a) we see that in I_1 we may take

$$\frac{du}{dy} = -\frac{3U_m}{b^2}y$$

since $\overline{u'v'}$ is small. Equation (8) reduces to

$$I_1 = \frac{6\pi U_m}{b^2l^2}\int_{-b}^{b}\left(\beta\frac{d\alpha}{dy} - \alpha\frac{d\beta}{dy}\right)y\,dy.$$

If turbulent motion is just possible then equation (8) may be written

$$\frac{3}{2}K_1 = \frac{lb^3\displaystyle\int_{-b}^{b}\left\{\left(\frac{2\pi}{l}\right)^4(\alpha^2+\beta^2) + 2\left(\frac{2\pi}{l}\right)^2\left[\left(\frac{d\alpha}{dy}\right)^2+\left(\frac{d\beta}{dy}\right)^2\right] + \left(\frac{d^2\alpha}{dy^2}\right)^2 + \left(\frac{d^2\beta}{dy^2}\right)^2\right\}dy}{3\pi\displaystyle\int_{-b}^{b}\left(\beta\frac{d\alpha}{dy} - \alpha\frac{d\beta}{dy}\right)y\,dy} = b^3\Delta, \qquad (9a)$$

where $K_1 = 2\rho bU_m/\mu$ and the function α, β, of y and the length l are to be determined so that Δ is a minimum.

Assuming
$$\begin{cases} \alpha = \alpha_0 + \sum_1^\infty (-1)^{n+1}\alpha_n\cos\dfrac{n\pi y}{b}, \\ \beta = \beta_0 + \sum_1^\infty (-1)^{n+1}\beta_n\cos\dfrac{n\pi y}{b}, \end{cases}$$

which satisfy the boundary conditions if $\alpha_0 - \sum_1^\infty \alpha_n = \beta_0 - \sum_1^\infty \beta_n = 0$. Taking $\alpha_i = \beta_i = 0$ for $i > 2$, substituting the values α and β in Δ we have

$$\Delta = \frac{\begin{gathered}2b\left(\frac{2\pi}{l}\right)^4 (A^2 + P^2) + b\left(\frac{2\pi}{l}\right)^4 [(B^2 + Q^2) + (C^2 + R^2)] \\ + 2b\left(\frac{2\pi}{l}\right)^2 \left[\left(\frac{\pi}{b}\right)^2 (B^2 + Q^2) + \left(\frac{2\pi}{b}\right)^2 (C^2 + R^2)\right] \\ + b\left[\left(\frac{\pi}{b}\right)^4 (B^2 + Q^2) + \left(\frac{2\pi}{b}\right)^4 (C^2 + R^2\right]\end{gathered}}{\frac{2\pi}{l}\left[2b(AQ - BP) + 2b(AR - PC) + \frac{10b}{3}(CQ - BR)\right]},$$

where $\alpha_0 = A_1$, $\alpha_1 = B$, $\alpha_2 = C$, $\beta_0 = P$, $\beta_1 = Q$, $\beta_2 = $ R. Putting $A = B + C$ and $P = Q + R$ we have

$$\Delta = \frac{\frac{\lambda_1}{2}(B^2 + Q^2) + \frac{\lambda_2}{2}(C^2 + R^2) + \mu\,(BC + QR)}{\nu\,(CQ - BR)}$$

where $\lambda_1 = 2\left[3\left(\frac{2\pi}{l}\right)^4 + 2\left(\frac{2\pi}{l}\right)^2\left(\frac{\pi}{b}\right)^2 + \left(\frac{\pi}{b}\right)^4\right]$, $\nu = \frac{20\pi}{3l}$,

$$\lambda_2 = 2\left[3\left(\frac{2\pi}{l}\right)^4 + 2\left(\frac{2\pi}{l}\right)^2\left(\frac{2\pi}{b}\right)^2 + \left(\frac{2\pi}{b}\right)^4\right], \quad \mu = 4\left(\frac{2\pi}{l}\right)^4. \quad (9b)$$

The conditions for a minimum value of Δ are

$$\begin{cases} B\lambda_1 + C\mu + R\nu\Delta = 0 \\ C\lambda_2 + B\mu - Q\nu\Delta = 0 \\ Q\lambda_1 + R\mu - C\nu\Delta = 0 \\ R\lambda_2 + Q\mu + B\nu\Delta = 0 \end{cases}$$

which have a solution provided the determinant

$$\begin{vmatrix} \lambda_1 & \mu & 0 & \Delta\nu \\ \mu & \lambda_2 & -\Delta\nu & 0 \\ 0 & -\Delta\nu & \lambda_1 & \mu \\ \Delta\nu & 0 & \mu & \lambda_2 \end{vmatrix} \equiv (\lambda_1\lambda_2 - \mu^2 + \Delta^2\nu^2)^2 = 0.$$

Substituting the value of λ_1, λ_2, μ, ν, from equations (9b) and minimizing this expression with respect to l we find $K = 131.4$. This is a value of the criterion when only three terms are used in the series α and β.

If we take $\alpha_i = \beta_i = 0$ for $i > 3$ we have

$$\alpha = A + B\cos\frac{\pi y}{b} - C\cos\frac{2\pi y}{d} + D\cos\frac{3\pi y}{b},$$

$$\beta = P + Q\cos\frac{\pi y}{b} - R\cos\frac{2\pi y}{b} + S\cos\frac{3\pi y}{b}.$$

Substituting these values in Δ we arrive at a determinant of the 6th order. This determinant is similar to that of the fourth order in the previous calculation and is also a perfect square. In fact, it can be easily proven that all succeeding determinants are perfect squares.

From the determinant of the 6th order we obtain a value for $K_1 = 120.9$ and from the 8th order $K_1 = 118.5$. Due to the mathematical computation involved this process becomes very laborious.

In the preceding work we expressed α and β by Fourier's Series which satisfied the boundary conditions exactly and determined conditions on the coefficients which were necessary for a minimum value of Δ and so found an approximate minimum for K_1. We can also determine by Calculus of Variations, differential equations which α and β must satisfy, if they furnish a minimum value of Δ, solve these equations and determine the arbitrary constants so that the boundary conditions are satisfied.

Beginning with the expression for Δ in (9) by the ordinary process of Calculus of Variations[4] we arrive at the differential equations,

$$2\left[\left(\frac{2\pi}{l}\right)^4\alpha - 2\left(\frac{2\pi}{l}\right)^2\frac{d^2\alpha}{dy^2} + \frac{d^4\alpha}{dy^4}\right] = -\Delta\left(\beta + 2y\frac{d\beta}{dy}\right)$$

$$2\left[\left(\frac{2\pi}{l}\right)^4\beta - 2\left(\frac{2\pi}{l}\right)^2\frac{d^2\beta}{dy^2} + \frac{d^4\beta}{dy^4}\right] = \Delta\left(\alpha + 2y\frac{d\alpha}{dy}\right).$$

Putting $\dfrac{\Delta}{2\left(\frac{2\pi}{l}\right)^4} = k_1$, and $y' = \dfrac{2\pi}{l}y$ and omitting ($'$), we have

$$\begin{aligned}\alpha - 2\frac{d^2\alpha}{dy^2} + \frac{d^4\alpha}{dy^4} &= -k_1\left(\beta + 2y\frac{d\beta}{dy}\right)\\ \beta - 2\frac{d^2\beta}{dy^2} + \frac{d^4\beta}{dy^4} &= k_1\left(\alpha + 2y\frac{d\alpha}{dy}\right).\end{aligned} \tag{10}$$

Multiplying the second equation in (10) by i adding it to the first and replacing ik by k, we have

$$(D^2 - 1)^2(\alpha + i\beta) = k(1 + 2yD)(\alpha + i\beta). \tag{11}$$

This equation has four linearly independent solutions of the form

$$\alpha + i\beta = P + kQ + k^2R + \dots$$

where $P = \cosh y$, $\sinh y$, $y \cosh y$ or $y \sinh y$. The general solution contains four arbitrary complex constants. The boundary conditions which the arbitrary constants must satisfy separate them into two sets one corresponding to the solutions with $P = \cosh y$ and $y \sinh y$, the other to the solutions with $P = \sinh y$ and $y \cosh y$. The former are found to lead to a lower value of k. Assuming a power series in y for $\alpha + i\beta$ and substituting in (11) we find with $P = \cosh y$.

$$\alpha_1 + i\beta_1 = \begin{cases} 1 + \frac{y^2}{2!} + (1+k)\frac{y^4}{4!} + (1+7k)\frac{y^6}{6!} + (1 + 22k + 9k^2)\frac{y^8}{8!} + \\ (1 + 50k + 109k^2)\frac{y^{10}}{10!} + (1 + 95k + 583k^2 + 153k^3)\frac{y^{12}}{12!} + \\ (1 + 161k + 2097k^2 + 2595k^3)\frac{y^{14}}{14!} + (1 + 252k + 6006k^2 + \\ 19612k^3 + 3825k^4)\frac{y^{16}}{16!} + (1 + 372k + 14574k^2 + 97732k^3 + \\ 82905k^4)\frac{y^{18}}{18!} + \ldots \end{cases}$$

and with $P = \dfrac{y \sinh y}{2}$

$$\alpha_2 + i\beta_2 = \begin{cases} \frac{y^2}{2!} + \frac{2y^4}{4!} + (3+5k)\frac{y^6}{6!} + (4 + 28k)\frac{y^8}{8!} + (5 + 90k + \\ 65k^2)\frac{y^{10}}{10!} + (6 + 220k + 606k^2)\frac{y^{12}}{12!} + (7 + 455k + 3037k^2 + \\ 1365k^3)\frac{y^{14}}{14!} + (8 + 840k + 10968k^2 + 17880k^3)\frac{y^{16}}{16!} + \\ (9 + 1428k + 32094k^2 + 122468k^3 + 39585k^4)\frac{y^{18}}{18!} + \ldots \end{cases}$$

where the coefficients are determined by the law

$$A_{n+4} - 2A_{n+2} + \{1 - (2n+1)k\}A_n = 0.$$

Taking the solution $\alpha + i\beta = (C_1 + iC_2)(\alpha_1 + iB_1) + (C_3 + iC_4)(\alpha_2 + i\beta_2)$,

then
$$\begin{aligned} \alpha &= C_1\alpha_1 - C_2\beta_1 + C_3\alpha_2 - C_4\beta_2 \\ \beta &= C_1\beta_1 + C_2\alpha_1 + C_3\beta_2 + C_4\alpha_2. \end{aligned}$$

The arbitrary constants C_1, C_2, C_3, C_4 must satisfy the conditions given by $\alpha = 0$, $\beta = 0$, $\frac{\partial\alpha}{\partial y} = \alpha' = 0$, $\frac{\partial\beta}{\partial y} = \beta' = 0$ when $y = \frac{2\pi b}{l}$. Eliminating C_1, C_2, C_3, C_4,

$$\begin{vmatrix} \alpha_1 & -\beta_1 & \alpha_2 & -\beta_2 \\ \beta_1 & \alpha_1 & \beta_2 & \alpha_2 \\ \alpha_1' & -\beta_1' & \alpha_2' & -\beta_2' \\ \beta_1' & \alpha_1' & \beta_2' & \alpha_2' \end{vmatrix} = 0.$$

The left-hand member is of the form $R^2 + S^2$ where

$$R + iS = \begin{vmatrix} \alpha_1 + i\beta_1 & \alpha_2 + i\beta_2 \\ \alpha_1' + i\beta_1' & \alpha_2' + i\beta_2' \end{vmatrix} \text{ and } R = S = 0. \qquad (12)$$

When the values of α_1, α_2, β_1, β_2, are substituted in (12) S is found to vanish identically and $R = 0$ gives the following equation:

$$0 = 1 + \frac{2y^2}{3!} + \frac{8y^4}{5!} + \frac{32y^6}{7!} + (128 + 32k^2)\frac{y^8}{9!} + (512 + 320k^2)\frac{y^{10}}{11!} + (2048 + 2816k^2)\frac{y^{12}}{13!} + (8192 + 21504k^2)\frac{y^{14}}{15!} + (32768 + 147456k^2 + 15360k^4)\frac{y^{16}}{17!} + (131072 + 933888k^2 + 276480k^4)\frac{y^{18}}{19!} + \dots$$

where $y = \dfrac{2\pi b}{l}$. We wish to determine the value of y which gives the least value of $K_1 = \dfrac{4}{3}\left(\dfrac{2\pi}{l}\right)^4 b^3k$. We therefore assume a series for $\dfrac{32k^2y^6}{9!}$ of the form

$$\frac{A}{y^2} + B + Cy^2 + Dy^4 + Ey^6.$$

Substituting and determining the coefficients A, B, C, D, E, we find

$$\frac{32k^2y^6}{9!} = -\frac{1.00556}{y^2} - 0.244949 - 0.04020465y^2 - 0.001426054y^4 + 0.00010464y^6.$$

which gives a minimum value of $K = 116.84$.

[1] Collected Papers, **2**, p. 524.

[2] Transactions, *Amer. Math. Soc.*, 1905.

[3] *Proc. Roy. Irish Soc.*, **27**, 1907.

[4] Wilson, *Advanced Calculus*, p. 400.